Hale School

Mathematics Specialist

Test 5 --- Term 3 2017

Applications of Differentiation and Modelling Motion

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Instructions:

- **Calculators are allowed**
- **External notes are not allowed**
- **Duration of test: 45 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

Question 1 (4 marks)

Given the equation, $\sqrt{xy} = \ln(\sin y + 2)$ determine $\frac{dy}{dx}$.

$$\frac{x\frac{dy}{dx} + y}{2\sqrt{xy}} = \frac{(\cos y)\frac{dy}{dx}}{\sin y + 2}$$

✓ diff $\sqrt{xy}$ correctly

✓ diff $\ln(\sin y + 2)$ correctly.

$$\left(x\frac{dy}{dx} + y\right)(\sin y + 2) = 2\cos y\sqrt{xy}\,\frac{dy}{dx}$$

✓ Expands

$$\frac{dy}{dx}\left(x(\sin y + 2)\right) + y(\sin y + 2) = 2\cos y\sqrt{xy}\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y(\sin y + 2)}{2\cos y\sqrt{xy} - x(\sin y + 2)}$$

✓ states in terms of $\frac{dy}{dx}$

Question 2 (5 marks)

Find the general solution to the following differential equations:

(a) $\frac{dy}{dx} = \cos^2 y$ (2 marks)

$$\int \frac{1}{\cos^2 y}\, dy = \int 1 \, dx$$ ✓ separates variables

$$\tan y = x + C$$ ✓ Integrates to determine general $sol^{\underline{n}}$

(b) $(x^2 - 1)^2 \frac{dy}{dx} = \frac{2x}{3y}$ (3 marks)

$$\int 3y \, dy = \int \frac{2x}{(x^2-1)^2}\, dx$$ ✓ separates variables

$$\frac{3y^2}{2} = -\frac{1}{(x^2-1)} + c$$ ✓ integrates $\frac{2x}{(x^2-1)^2}$ correctly

$$\frac{3y^2 - 2c}{2} = -\frac{1}{(x^2-1)}$$ ✓ integrates $3y$ correctly

$$(x^2-1)(2c - 3y^2) = 2$$

Question 3 (3 marks)

The differential equation for a curve passing through the point (1, -1) is given by $\frac{dy}{dx} = xy - x^2$. Use the incremental formula $\delta y = \frac{dy}{dx} \times \delta x$, with $\delta x = 0.2$, to calculate an estimate for the y-coordinate of the curve when $x = 1.4$.

- $n=0 \quad x_0 = 1 \quad y_0 = -1$

- $n=1$, $x_1 = 1.2$

$$y_1 = y_0 + \delta y$$
$$= -1 + \frac{dy}{dx}\delta x$$
$$= -1 + \left[(1)(-1) - (1)^2\right]0.2$$
$$= -1 - 0.4$$
$$= -1.4$$

✓ Increases x value by 0.2.

✓ uses incremental formula to determine y_1

- $n=2$, $x_2 = 1.4$

$$y_2 = y_1 + \delta y$$
$$= -1.4 + \left[(1.2)(-1.4) - (1.2)^2\right]0.2$$
$$= -1.4 + \left[-3.12\right] \times 0.2$$
$$= -2.024$$

✓ uses incremental formula to determine y_2

Question 4 (7 marks)

An object undergoing SHM is defined by the differential equation $\frac{dv}{dt} = -n^2 x$.

(a) Given $\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$, use integration techniques to show that $v^2 = n^2(A^2 - x^2)$ where A is the amplitude of the motion. (4 marks)

$\frac{dv}{dx} \times \frac{dx}{dt} = -n^2 x$

$v \frac{dv}{dx} = -n^2 x$ ✓ recognises $v = \frac{dx}{dt}$

$\int v \, dv = \int -n^2 x \, dx$ ✓ separates variables

$\frac{v^2}{2} = -\frac{n^2 x^2}{2} + c$ (when $v=0$ $x=A$) ✓ recognises $v=0$ $x=A$ for SHM

$0 = \frac{-n^2 A^2}{2} + c$

$c = \frac{n^2 A^2}{2}$ ✓ Determines c

$\therefore v^2 = -n^2 x^2 + n^2 A^2$

$v^2 = n^2(A^2 - x^2)$

(b) If $x = 4\ m$ when $v = -18\ m/s$ and $x = -3\ m$ when $v = 24\ m/s$, find the period and amplitude of the motion. (3 marks)

$(-18)^2 = n^2(A^2 - 4^2)$ ... (1)

$(24)^2 = n^2(A^2 - (-3)^2)$ ... (2) ✓ establishes equations

Solving simultaneously -

$n = 6$, $A = 5\ m$ ✓ solves for n and A.

$T = \frac{\pi}{3}$ secs. ✓ evaluates period.

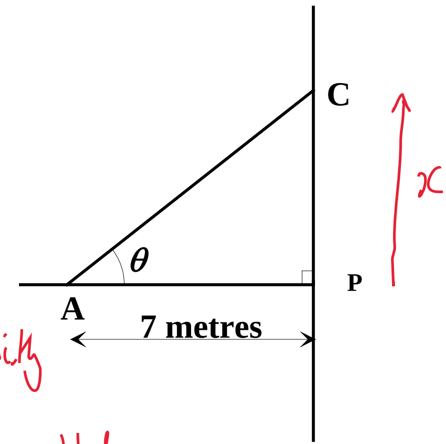
Question 5 (8 marks)

A rotating radar gun, at A, for measuring the speed of cars is positioned on a straight section of Great Eastern Highway. The radar gun is set up 7 metres from P, the nearest point on the side of the road, as shown in the diagram. $\angle APC = \frac{\pi}{2}$ and $\angle PAC = \theta^{r}$.

The gun is tracking the car C automatically. It is rotating at 0.358 radians per second at the instant the car is 24 m from P.

(a) Show that the car is exceeding the speed limit of 110 km/hr. (5 marks)

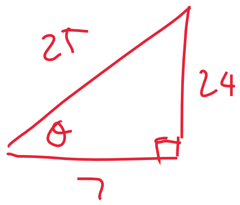
$x = 24 \qquad \frac{dx}{dt} = ? \qquad \frac{d\theta}{dt} = 0.358$ r/sec



$\tan\theta = \frac{x}{7}$ ✓ correct trig relationship

$7\tan\theta = x$

$\frac{7}{\cos^2\theta}\frac{d\theta}{dt} = \frac{dx}{dt}$ ✓ diff implicitly



$\frac{7}{\left(\frac{7}{25}\right)^2}(0.358) = \frac{dx}{dt}$ ✓ substitutes values.

$31.96_{m/s} = \frac{dx}{dt}$

$\frac{dx}{dt} = 115.07$ km/h ✓ correct result

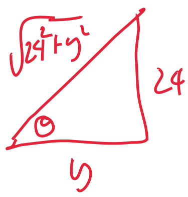
$115.07 > 110 \quad \therefore$ exceeding speed limit. ✓ shows speed limit exceeded

(b) The motorist wants to challenge the fine imposed. After viewing photographic evidence, he suspects that the distance AP was greater than 7 metres. What distance would AP need to be to ensure that the motorist has not exceeded the speed limit.

(3 marks)

✓ determines $\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{110}{3.6} = 30.5\dot{5}$ m/s $\qquad AP = y$



✓ establishes equation

$\frac{dx}{dt} = \frac{y}{\cos^2\theta}\frac{d\theta}{dt}$

✓ determines AP

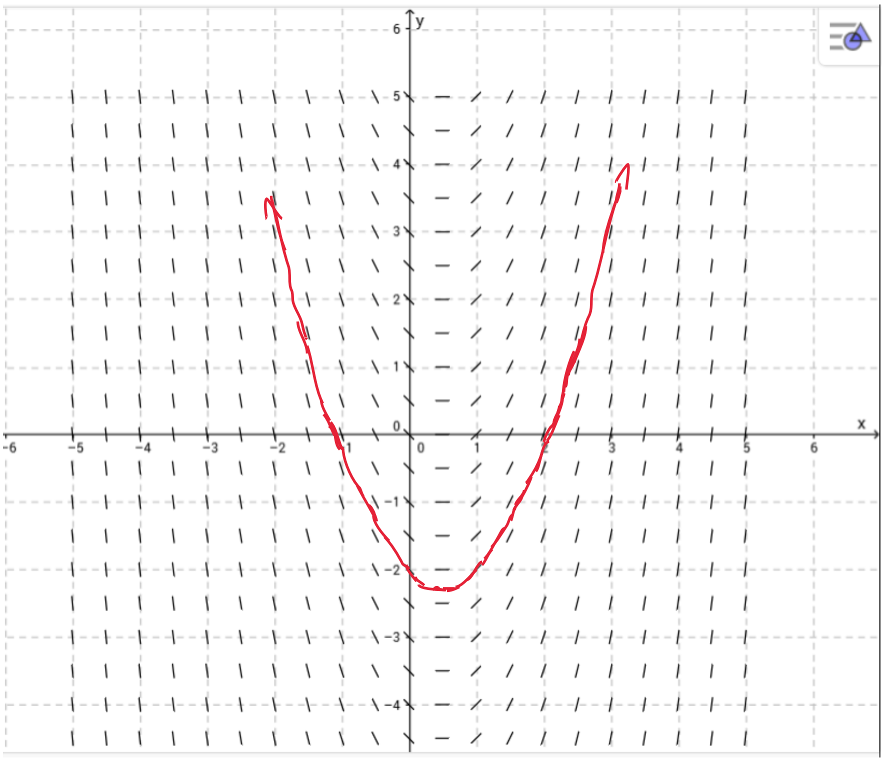
$30.\dot{5} = \frac{y}{\left(\frac{y}{\sqrt{24^2+y^2}}\right)^2} 0.358$

$\therefore y = 7.39$ m

$AP > 7.39$ m

Question 6 (7 marks)

The diagram below shows the slope field of a differential equation $\frac{dy}{dx} = f(x, y)$



(a) Determine the general differential equation that would yield this slope field.

(3 marks)

$y = a(x - \frac{1}{2})^2 + c$

$\frac{dy}{dx} = 2a(x - \frac{1}{2})$

$\frac{dy}{dx} = a(2x - 1)$

✓ states solution as a differential equation

✓ recognises d.e. as linear

✓ Identifies $x = \frac{1}{2}$ as T.P.

(b) On the slope field given above, draw in a curve representing the particular solution with initial condition (0,-2) .

(2 marks)

✓ T.P. ~ $(\frac{1}{2}, -2.25)$ ✓ x-int ~ $(-1.1, 0)$ $(2.1, 0)$

(c) Determine the equation for the particular solution if dy/dx=-1 at the point (0,-2) .

(2 marks)

$-1 = 2a(0 - \frac{1}{2})$

$a = 1$

✓ determines a

$\therefore \quad y = x^2 - x + c$ $\quad c = -2$

$y = x^2 - x - 2$ ✓ correct p.s.

Question 7 (8 marks)

(a) Show that if $P = \dfrac{a}{b + ke^{-at}}$, where $a,$ b and k are positive constants, then $\dfrac{dP}{dt} = aP - bP^2$. (4 marks)

$$P = \frac{a}{b + ke^{-at}}$$

$$\frac{dP}{dt} = \frac{a}{(b+ke^{-at})^2} \times (ake^{-at})$$ ✓ differentiates

$$= \frac{a}{(b+ke^{-at})} \times \frac{ake^{-at}}{(b+ke^{-at})}$$

$$= P \times \left(\frac{ake^{-at}}{b+ke^{-at}} \right)$$ ✓ factor out P

$$= P \times \left(\frac{a(b+ke^{-at}) - ab}{b+ke^{-at}} \right)$$ ✓ separates terms

$$= P \times \left(a - \frac{ab}{b+ke^{-at}} \right)$$

$$= P \times (a - bP)$$ ✓ shows result

$$= aP - bP^2$$

(b) In Dwellingup, it was initially discovered that 10 trees were infected with dieback disease. The resultant growth in the dieback disease is modelled by the equation $\frac{dP}{dt} = 0.1P - 0.000025P^2$, where P is the number of infected trees t months after the initial discovery of the disease.
Use your result from (a) to express P as a function of t. (2 marks)

$$\frac{dP}{dt} = 0.1P - 0.000025P^2$$

$$P = \frac{0.1}{0.000025 + ke^{-0.1t}}$$

✓ transposes values correctly

$$10 = \frac{0.1}{0.000025 + k}$$

$$k = 0.01 - 0.000025$$

$$k = 0.009975$$

✓ finds k value.

$$\therefore P = \frac{0.1}{0.000025 + 0.009975e^{-0.1t}}$$

(c) Calculate the number of months taken for the number of trees infected by dieback to reach 80% of its limiting value. (2 marks)

$t \to \infty \quad P \to 4000$

$0.8P = 3200$

✓ finds 80% of limiting value

$$3200 = \frac{0.1}{0.000025 + 0.009975e^{-0.1t}}$$

$t = 73.75$ months ✓ calculates month.

END OF TEST